MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 128
OCTOBER 1974, PAGES 973–979

On the Eigenvectors of a Finite-Difference Approximation to the Sturm-Liouville Eigenvalue Problem

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Abstract. This paper is concerned with a centered finite-difference approximation to to the nonselfadjoint Sturm-Liouville eigenvalue problem

$$L[u] = -[a(x)u_x]_x - b(x)u_x + c(x)u = \lambda u, \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

It is shown that the eigenvectors W_p of the $M \times M$-matrix ($\Delta x = 1/(M+1)$ mesh size), which approximates L, are bounded in the maximum norm independent of M if they are normalized so that $|W_p|_2 = 1$.

1. Introduction. The present paper is concerned with the nonselfadjoint problem

$$(1) \qquad \begin{aligned} -[a(x)u_x]_x - b(x)u_x + c(x)u = \lambda u, \qquad 0 < x < 1,\\ u(0) = u(1) = 0, \end{aligned}$$

where $a(x) \geqslant \alpha > 0$, $c(x) \geqslant 0$, and a, b, c are all bounded and smooth functions. This problem has an infinite sequence of positive and distinct eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and a corresponding sequence of smooth eigenfunctions $u^1, u^2, u^3, \cdots$ (see, for instance Protter-Weinberger [10, p. 37] and Coddington-Levinson [4, p. 212]). Following Courant-Hilbert [5, p. 334], the eigenfunctions u^p are uniformly bounded in the supremum norm if they are normalized so that

$$\int_0^1 |u^p(x)|^2\, dx = 1, \qquad p = 1, 2, 3, \cdots .$$

Of course, by the well-known transformation

$$(2) \qquad u(x) = \exp\left(-\tfrac{1}{2}\int_0^x \frac{b(t)}{a(t)}\, dt\right) w(x)$$

(1) may be put in the selfadjoint form

Received October 4, 1973.
AMS (MOS) subject classifications (1970). Primary 65L15.
Key words and phrases. Sturm-Liouville eigenvalue problem, finite-difference approximation.

$$- [a(x)w_x]_x + \tilde{c}(x)w = \lambda w, \qquad 0 < x < 1,$$
$$w(0) = w(1) = 0,$$

where

$$\tilde{c}(x) = c(x) + \tfrac{1}{2}b_x(x) + \tfrac{1}{4}b^2(x)/a(x).$$

Here, in order to obtain $\tilde{c}(x) \geq 0$, we have to make a restricting assumption on b_x. Therefore, we choose the direct approximation of (1) by means of the finite-difference equations

$$(3) \qquad -\frac{a_{k+1/2}(v_{k+1} - v_k) - a_{k-1/2}(v_k - v_{k-1})}{\Delta x^2} - b_k \frac{v_{k+1} - v_{k-1}}{2\Delta x} + c_k v_k = \Lambda v_k, \qquad k = 1, \cdots, M,$$
$$v_0 = v_{M+1} = 0,$$

where $M \in \mathbf{N}$, $\Delta x = 1/(M + 1)$, and $v_k = v(k\Delta x)$. Equivalently, we may write (3) in matrix-vector notation

$$(3') \qquad LV = \Lambda V,$$

where $V = (v_1, \cdots, v_M)^T$ and the matrix L may be easily derived from (3).

Let $|b(x)| \leq \beta$ and $0 < \Delta x < 2\alpha/\beta$. Then the matrix L is equivalent to a real symmetric matrix (see Carasso [2]). Using this fact and Theorem 1.8 of Varga [11], it can be shown that all eigenvalues Λ_p of (3) are real and positive, $0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \leq \Lambda_M$, and there exists a complete sequence of corresponding eigenvectors V^p. A result of Carasso [2, Corollary 1] says that there exist a constant K and an integer p_0, both independent of M, such that

$$|V^p|_\infty = \max_{1 \leq k \leq M} |v_k^p| \leq K p^{1/2}, \qquad p_0 \leq p \leq M,$$

if

$$|V^p|_2^2 = \Delta x \sum_{k=1}^{M} |v_k^p|^2 = 1.$$

In the selfadjoint case, this result goes back to Bückner [1]. In this paper, we prove the following theorem:

THEOREM. *Let* $a(x) \geq \alpha > 0$ *and* $c(x) \geq 0$, $0 < x < 1$. *Assume that* a, b, *and* c *are differentiable bounded functions with bounded derivatives. Let* $|b(x)| \leq \beta$. *Let* $0 < \Delta x \leq \alpha/\beta$ *and let* $\{V^p\}_{p=1}^M$ *be the eigenvectors of* (3), *normalized so that* $|V^p|_2 = 1$. *Then*

$$|V^p|_\infty \leq \kappa, \qquad p = 1, \cdots, M,$$

for some constant κ *independent of* M.

Remark 1. In the case of the equation $u_{xx} = \lambda u$, this result may be proved by explicit computation of the eigenvectors V^p (see Isaacson-Keller [9, 9.1.1]).

Applications of the Theorem to the theory of finite-difference approximations to parabolic and hyperbolic partial differential equations are given in [3], [7], [8].

2. Proof of the Theorem. Instead of L, we consider, as in [2], the eigenvectors $D^{-1}V^p$ of the similar matrix $D^{-1}LD$ defined below. But, in contrast to Carasso [2], who uses a discrete maximum principle for his estimation, we then transpose the proof of Courant-Hilbert [5, p. 334] to the resulting discrete problem.

The following basic results are needed.

LEMMA 1 (CARASSO [2, LEMMA 1], [3, LEMMA 3.1]). *Let* $D = (d_1, \cdots, d_M)$ *be the diagonal matrix with*

$$d_1 = 1, \quad d_i = + \left[\prod_{k=1}^{i-1} \frac{a_{k+1/2} - b_{k+1}\Delta x/2}{a_{k+1/2} + b_k \Delta x/2} \right]^{1/2}, \quad i = 2, \cdots, M.$$

For $0 < \Delta x < 2\alpha/\beta$, we have $d_i > 0$ and

$$|D|_\infty \leqslant \kappa_1, \qquad |D^{-1}|_\infty \leqslant \kappa_2$$

for constants κ_1, κ_2 independent of M. Furthermore, $D^{-1}LD = (P + Q)/\Delta x^2$ where $P = (p_{ik})_{i,\,k=1,\cdots,M}$,

$$\begin{aligned} p_{ik} &= (p_{k+1/2} + p_{k-1/2}), && i = k, \\ &= -p_{k+1/2}, && i = k+1, \\ &= -p_{i+1/2}, && k = i+1, \\ &= 0, && \text{otherwise}, \end{aligned}$$

$$p_{k+1/2} = (a_{k+1/2} - b_{k+1}\Delta x/2)^{1/2}(a_{k+1/2} + b_k \Delta x/2)^{1/2},$$

and $Q = (q_1, \cdots, q_M)$ is the diagonal matrix with

$$q_k = (a_{k+1/2} + a_{k-1/2}) - (p_{k+1/2} + p_{k-1/2}) + \Delta x^2 c_k.$$

Remark 2. The change of variables $V = DW$ is a discrete analog to (2) [2].

LEMMA 2 (CARASSO [2, THEOREM 1]). *Let* Λ_p, V^p *be the characteristic pairs of the matrix* L *with* $|V^p|_2 = 1$. *Let* u^p *be an eigenfunction of* (1) *corresponding to* λ_p *and let* U^p *be the vector of dimension* M *obtained from* u^p *by mesh-point evaluation. Assume* u^p *normalized so that* $|D^{-1}U^p|_2 = |D^{-1}V^p|_2$, *then, as* $\Delta x \longrightarrow 0$, *we have*

$$|\lambda_p - \Lambda_p| \leqslant \kappa_3(p)\Delta x^2, \qquad |U^p - V^p|_2 \leqslant \kappa_4(p)\Delta x^2, \tag{4}$$

where κ_3, κ_4 *are positive constants depending only on* p.

In the selfadjoint case, Lemma 2 was proved by Gary [6].

Remark 3. The estimation (4) implies $|U^p - V^p|_\infty \leqslant \kappa_4(p)\Delta x^{3/2}$.

LEMMA 3. *Let*

$$C_l(W) = \sum_{k=1}^{l} [p_{k+1/2}(w_k - w_{k+1}) - p_{k-1/2}(w_k - w_{k-1})]\, q_k w_k/\Delta x^2.$$

Then, under the assumptions of the Theorem,

$$C_l(W^p) = -q_l p_{l+1/2} w_{l+1}^p w_l^p/\Delta x^2 + O(1), \qquad l = 1, \cdots, M,$$

where $W^p = D^{-1}V^p$ *and* $O(1)$ *denotes a function which has a bound independent of* M.

Proof. We show at first that $|q_k/\Delta x^2| \leqslant \kappa_5$ independently of M. To this end, it suffices to consider $a_{k+1/2} - p_{k+1/2}$. By means of the binomial theorem, we obtain

$$a_{k+1/2} - p_{k+1/2}$$

$$= a_{k+1/2} - a_{k+1/2}\left(1 - \frac{b_{k+1}\Delta x}{4a_{k+1/2}} + O(\Delta x^2)\right)\left(1 + \frac{b_k \Delta x}{4a_{k+1/2}} + O(\Delta x^2)\right)$$

Inserting $b_{k+1} = b_k + O(\Delta x)$, we find that

$$a_{k+1/2} - p_{k+1/2} = O(\Delta x^2). \tag{5}$$

Now, since $w_0^p = 0$,

$$C_l(W^p) = -q_l p_{l+1/2} w_{l+1}^p w_l^p/\Delta x^2$$

$$+ \sum_{k=1}^{l} \frac{q_k}{\Delta x^2}(p_{k+1/2} - p_{k-1/2}) w_k^p w_k^p + \sum_{k=1}^{l-1} \frac{q_{k+1} - q_k}{\Delta x^2} p_{k+1/2} w_{k+1}^p w_k^p.$$

But, by the mean value theorem, we have $p_{k+1} - p_k = O(\Delta x)$ and $(q_{k+1} - q_k)/\Delta x^2 = O(\Delta x)$. Hence, using Schwarz's inequality and $|W^p|_2 \leqslant \kappa_6$, we obtain the desired result.

Now, according to Lemma 1, it suffices to prove the Theorem for the eigenvectors $W^p = D^{-1}V^p$ of the matrix $(P + Q)/\Delta x^2$ which also has the eigenvalues Λ_p. We multiply the kth row of

$$(1/\Delta x^2)(P + Q)W^p = \Lambda_p W^p$$

by

$$p_{k+1/2}(w_k^p - w_{k+1}^p) - p_{k-1/2}(w_k^p - w_{k-1}^p)$$

and obtain, by adding all rows from $k = 1$ to $k = l$,

$$\left[\frac{p_{l+1/2}(w_l^p - w_{l+1}^p)}{\Delta x}\right]^2 + C_l(W^p)$$

$$+ \Lambda_p p_{l+1/2} w_{l+1}^p w_l^p - \Lambda_p \Delta x \sum_{k=1}^{l} p_x(\xi_k) w_k^p w_k^p = \left[\frac{p_{1/2}(w_1^p - w_0^p)}{\Delta x}\right]^2, \tag{6}$$

where $(k - 1/2)\Delta x < \xi_k < (k + 1/2)\Delta x$. In order to eliminate the term on the right side of (6), we sum up Eqs. (6) for $l = 1$ to $l = M$, add $(p_{1/2}(w_1^p - w_0^p)/\Delta x)^2$ to both sides, and divide by $M + 1 = 1/\Delta x$. Then

$$\left[\frac{p_{1/2}(w_1^p - w_0^p)}{\Delta x}\right]^2 = \Delta x \sum_{l=0}^{M} \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x}\right]^2 + \Delta x \sum_{l=1}^{M} C_l(W^p)$$

$$+ \Lambda_p \Delta_x \sum_{l=1}^{M} p_{l+1/2} w_{l+1}^p w_l^p - \Lambda_p \Delta x^2 \sum_{l=1}^{M} \sum_{k=1}^{l} p_x(\xi_k) w_k^p w_k^p. \tag{7}$$

But

$$0 < \alpha/2 < p_{l+1/2} < \kappa_7 \quad (\kappa_7 > 1) \tag{8}$$

if $0 < \Delta x \leqslant \alpha/\beta$. Thus, using the fundamental relation

$$\sum_{l=0}^{M} p_{l+1/2}(w_{l+1} - w_l)^2 = W^T P W$$

$(w_0 = w_{M+1} = 0)$, we derive

$$\Delta x \sum_{l=0}^{M} \left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x}\right]^2 \leqslant \Delta x \kappa_7 (W^p)^T P W^p / \Delta x^2$$

$$= \Delta x \kappa_7 (W^p)^T (P + Q) W^p / \Delta x^2 - \Delta x \kappa_7 \sum_{l=1}^{M} q_l w_l^p w_l^p / \Delta x^2$$

$$\leqslant \Delta x \kappa_7 \Lambda_p (W^p)^T W^p + \kappa_5 \kappa_7 \Delta x (W^p)^T W^p = \kappa_7 \Lambda_p + \kappa_5 \kappa_7,$$

since $|q_l/\Delta x^2| \leqslant \kappa_5$ independently of M and $|W^p|_2^2 = 1$. Hence, applying Schwarz's inequality, we find from (7) by means of the assumption and Lemma 3 that

$$[p_{1/2}(w_1^p - w_0^p)/\Delta x]^2 \leqslant \kappa_8 \Lambda_p + \kappa_9.$$

From this estimation, Eq. (6), and Lemma 3, we deduce that

$$(\Lambda_p - q_l/\Delta x^2)p_{l+1/2}w_{l+1}^p w_l^p \leqslant \kappa_{10}\Lambda_p + \kappa_{11}.$$

Consequently, observing (8), we obtain, in case $\Lambda_p > \kappa_5$, that

$$w_{l+1}^p w_l^p \leqslant \kappa_{12}, \qquad l = 1, \cdots, M, \tag{9}$$

for some constant κ_{12} independent of M. For $\Lambda_p \leqslant \kappa_5$, the assertion of the Theorem follows by Lemma 2, Remark 3.

Finally, we return once more to Eq. (6). The above estimation yields

$$\left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x}\right]^2 \leqslant \kappa_{13}\Lambda_p + \kappa_{14}, \qquad l = 1, \cdots, M,$$

or, using (9),

$$\begin{aligned} |w_l^p|^2 &\leqslant |w_l^p|^2 + |w_{l+1}^p|^2 = (w_{l+1}^p - w_l^p)^2 + 2w_l^p w_{l+1}^p \\ &= \frac{\Delta x^2}{p_{l+1/2}^2}\left[\frac{p_{l+1/2}(w_{l+1}^p - w_l^p)}{\Delta x}\right]^2 + 2w_{l+1}^p w_l^p \\ &\leqslant \Delta x^2(\kappa_{15}\Lambda_p + \kappa_{16}) + \kappa_{17} = \kappa, \qquad l = 1, \cdots, M. \end{aligned}$$

Hence, $\max_{1 \leqslant p \leqslant M}\{|W^p|_\infty\}$ is bounded independently of M.

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